Q1

1

Find an expression for $\frac{dy}{dx}$ in terms of x and y for the following

(i) $e^{xy} + \ln(xy) = \operatorname{cosec}(x) + 4$

(ii) $4\cos(x^2y) - 3e^{x^2y} = 4e^y$

(alternatively, rewrite $\ln(xy)$ as $\ln x + \ln y$ before differenciating)

[5]

i) $\frac{d}{dx}$ $\quad \frac{d(xy)}{dx} e^{xy} + \frac{d(xy)}{dx}\frac{1}{xy} = -\operatorname{cosec}x\cot x + 0$

$\left(x\frac{dy}{dx} + y\right)e^{xy} + \left(x\frac{dy}{dx} + y\right)\frac{1}{xy} = -\operatorname{cosec}x\cot x$

$\frac{dy}{dx}\left(xe^{xy} + \frac{1}{y}\right) = -\operatorname{cosec}x\cot x - ye^{xy} - \frac{1}{x}$

$$\boxed{\frac{dy}{dx} = \frac{-\operatorname{cosec}x\cot x - ye^{xy} - \frac{1}{x}}{xe^{xy} + \frac{1}{y}}}$$

ii) $\frac{d}{dx}$ $\quad \frac{d(x^2y)}{dx}(-4\sin(x^2y)) + \frac{d(x^2y)}{dx}(-3e^{x^2y}) = 4e^y\frac{dy}{dx}$

$\left(x^2\frac{dy}{dx} + y2x\right)\left(-4\sin(x^2y) - 3e^{x^2y}\right) = 4e^y\frac{dy}{dx}$

$\frac{dy}{dx}\left(x^2(-4\sin(x^2y) - 3e^{x^2y}) - 4e^y\right)$

$= 2xy\left(4\sin(x^2y) + 3e^{x^2y}\right)$

$$\boxed{\frac{dy}{dx} = \frac{8xy\sin(x^2y) + 6xye^{x^2y}}{-4x^2\sin(x^2y) - 3x^2e^{x^2y} - 4e^y}}$$

Q2

2

Find the gradient at the point where $x = -2$ and y is an integer on the curve with equation $x^2y^2 - 5x = 22y$.

[5]

$\frac{d}{dx}$ $\quad x^2 2y\frac{dy}{dx} + y^2 2x - 5 = 22\frac{dy}{dx}$ ①

When $x = -2$

$(-2)^2y^2 - 5(-2) = 22y$

$4y^2 - 22y + 10 = 0$

$2y^2 - 11y + 5 = 0$

$(2y-1)(y-5) = 0$

$y = \frac{1}{2}, 5$

Since y must be an integer, $y = 5$.

Sub $x = -2$, $y = 5$ into ①

$(-2)^2 2(5)\frac{dy}{dx} + 5^2\times 2\times -2 - 5 = 22\frac{dy}{dx}$

$40\frac{dy}{dx} - 100 - 5 = 22\frac{dy}{dx}$

$18\frac{dy}{dx} = 105$

$\frac{dy}{dx} = \frac{105}{18} = \boxed{\frac{35}{6}}$

Q3

3

An ellipse has equation

$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

Find an expression for $\frac{dy}{dx}$ and hence show that the gradient of the ellipse at any point where it meets a line of the form $y = kx$ $(k \neq 0)$ is independent of x and y.

[4]

$\frac{d}{dx}$ $\quad \frac{2x}{4}+\frac{1}{9}2y\frac{dy}{dx}=0$

$$\frac{dy}{dx}=\frac{-\frac{2x}{4}}{\frac{2y}{9}}=-\frac{\cancel{2}x}{4}\times\frac{9}{\cancel{2}y}=\frac{-9x}{4y}$$

Sub in $y=kx$ to find the gradient where the ellipse meets the line:

$$\frac{dy}{dx}=\frac{-9\cancel{x}}{4k\cancel{x}}=\boxed{\frac{-9}{4k}}$$

The espression for $\frac{dy}{dx}$ does not involve (so is independant of) the values of x and y in the instances where the ellipse meets a line of the form $y=kx$ $(k\neq 0)$.

Q4

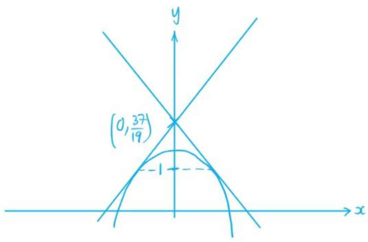
4

The curve C is described by the equation

$$\ln y + x^2y^2 = 9.$$

Show that the tangents of the two points on C where $y = 1$ meet at the point $\left(0, \frac{37}{19}\right)$.

[5]



$$y-y_1=m(x-x_1)$$

When $y=1$

$\ln 1 + x^2(1)^2 = 9$

$x^2=9$

$x=\pm 3$

$\therefore$ The two points are $(3, 1)$ and $(-3, 1)$

$\frac{d}{dx}$ $\quad \frac{1}{y}\frac{dy}{dx}+x^2 2y\frac{dy}{dx}+y^2 2x=0$

$$\frac{dy}{dx}\left(\frac{1}{y}+2x^2y\right)=-2xy^2$$

$$\frac{dy}{dx}=\frac{-2xy^2}{\frac{1}{y}+2x^2y}$$

At $(3, 1)$:

$$\frac{dy}{dx}=\frac{-2(3)1^2}{\frac{1}{1}+2(3)^2(1)}=-\frac{6}{19}$$

$$y-1=-\frac{6}{19}(x-3)$$

$$y=-\frac{6x}{19}+\frac{37}{19}$$

At $(-3, 1)$

$$\frac{dy}{dx}=\frac{-2(-3)(1)^2}{\frac{1}{1}+2(-3)^2(1)}=\frac{6}{19}$$

$$y-1=\frac{6}{19}(x-(-3))$$

$$y=\frac{6}{19}x+\frac{37}{19}$$

The y-intercept of both tangents is $\frac{37}{19}$, therefore the two tangents meet at the point $(0, \frac{37}{19})$.

Q5a

5a

The curve C is described by the equation

$$3x^2 + 2xy^3 + 16 = 0.$$

(a) Show that the normal to C at the point where $x = -4$ is parallel to the normal to C at the point where $x = 4$.

[4]

(b) Find the distance between the y-axis intercepts of these two normals.

[4]

a) For the two normals to be parallel, their gradients must be equal.

$$\text{gradient}_{(\text{at } x=-4)} = \text{gradient}_{(\text{at } x=4)}$$

$$\frac{d}{dx} \quad 6x + 2x\,3y^2\frac{dy}{dx} + y^3 2 + 0 = 0$$

$$\frac{dy}{dx}\left(6xy^2\right) = -6x - 2y^3$$

$$\frac{dy}{dx} = \frac{-6x-2y^3}{6xy^2} = \frac{-3x-y^3}{3xy^2}$$

when $x = 4$

$$3(4)^2 + 2(4)y^3 + 16 = 0$$

$$y^3 = -8$$

$$y = -2$$

$$\frac{dy}{dx} = \frac{-3(4)-(-2)^3}{3(4)(-2)^2} = \boxed{\frac{-1}{12}}$$

when $x = -4$

$$3(-4)^2 + 2(-4)y^3 + 16 = 0$$

$$y^3 = 8$$

$$y = 2$$

$$\frac{dy}{dy} = \frac{-3(-4)-8}{3(-4)(2)^2} = \boxed{\frac{-1}{12}}$$

We could use these tangent gradients to find the normal gradients, but it's easier to conclude...

The gradients of the tangents are the same, so the tangents are parallel, and therefore the normals must be parallel.

Q5b

5b

The curve C is described by the equation

$$3x^2 + 2xy^3 + 16 = 0.$$

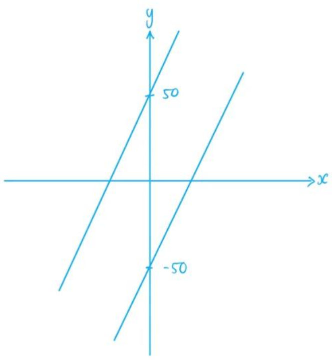
(a) Show that the normal to C at the point where $x = -4$ is parallel to the normal to C at the point where $x = 4$.

$\frac{dy}{dx} = \frac{-1}{12}$ where $(4, -2)$ and $(-4, 2)$

[4]

(b) Find the distance between the y-axis intercepts of these two normals.

[4]



b) $y - y_1 = m(x - x_1)$

$$m_n = \frac{-1}{\frac{dy}{dx}} = \frac{-1}{\frac{-1}{12}} = 12$$

Eqn of normal at $(4, -2)$:

$$y - (-2) = 12(x - 4)$$

$$y = 12x \boxed{-50}$$

Eqn of normal at $(-4, 2)$:

$$y - (2) = 12(x - (-4))$$

$$y = 12x \boxed{+50}$$

Find the difference between the two y-intercepts

$$\text{distance} = 50 - (-50) = \boxed{100}$$

Q6

6

Find the stationary points and determine their nature for the curve with equation $y^2 = 3x^2 - 2xy + 3$.

[8]

① $2y\frac{dy}{dx} = 6x - \left(2x\frac{dy}{dx} + y^2\right)$

$\frac{dy}{dx}(2y + 2x) = 6x - 2y$

$\frac{dy}{dx} = \frac{6x - 2y}{2y + 2x} = \boxed{\frac{3x - y}{x + y}} = 0$

$\frac{u}{v}$

② $3x - y = 0$

$y = 3x$

Sub $y = 3x$ into eqn for curve

$(3x)^2 = 3x^2 - 2x(3x) + 3$

$9x^2 = 3x^2 - 6x^2 + 3$

$12x^2 = 3$

$x^2 = \frac{1}{4}$ $\quad \therefore x = \sqrt{\frac{1}{4}} = \pm\frac{1}{2}$

When $x = \frac{1}{2}$

$y = 3\left(\frac{1}{2}\right) = \frac{3}{2}$

$\boxed{\left(\frac{1}{2}, \frac{3}{2}\right)}$

When $x = -\frac{1}{2}$

$y = 3\left(-\frac{1}{2}\right) = -\frac{3}{2}$

$\boxed{\left(-\frac{1}{2}, -\frac{3}{2}\right)}$

③ Classify stationary points

quotient rule

$\frac{d^2\left(\frac{u}{v}\right)}{dx^2} = \frac{(x+y)\left(3 - \cancel{\frac{dy}{dx}}^{\,0}\right) - (3x - y)\left(1 + \cancel{\frac{dy}{dx}}^{\,0}\right)}{(x+y)^2} = \frac{4y}{(+)}$

Since this is a squared number the denominator will always be +ve!

At $\left(\frac{1}{2}, \frac{3}{2}\right)$ $\quad 2^{nd}$ derivative: $\frac{4\left(\frac{3}{2}\right)}{(+)} > 0 \rightarrow$ $\boxed{\text{min point}}$

At $\left(-\frac{1}{2}, -\frac{3}{2}\right)$ $\quad 2^{nd}$ derivative: $\frac{4\left(-\frac{3}{2}\right)}{(+)} < 0 \rightarrow$ $\boxed{\text{max point}}$

Q7

7

The curve C is defined by

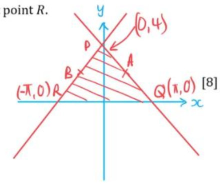
$$e^{\sin(xy)} = 1 \qquad \{y > 0\}$$

Points A and B have coordinates $\left(\frac{\pi}{2}, 2\right)$ and $\left(-\frac{\pi}{2}, 2\right)$ respectively.

The tangents to C at points A and B intersect at the point P.
The tangent to C at point A intersects the x-axis at point Q.
The tangent to C at point B intersects the x-axis at point R.

Find the area of triangle PQR.

[8]



$y - y_1 = m(x - x_1)$

$\frac{d}{dx}$ $\quad \boxed{\frac{d(\sin(xy))}{dx}} e^{\sin(xy)} = 0$

$\boxed{\frac{d(xy)}{dx}} \cos(xy) e^{\sin(xy)} = 0$

$\left(x\frac{dy}{dx} + y\right)\cos(xy)e^{\sin(xy)} = 0$

$x\frac{dy}{dx}\cancel{\left(\cos(xy)\right)e^{\sin(xy)}} = -y\cancel{\left(\cos(xy)\right)e^{\sin(xy)}}$

$\frac{dy}{dx} = -\frac{y}{x}$

At A, $x = \frac{\pi}{2}$, $y = 2$

$\frac{dy}{dx} = -\frac{2}{\frac{\pi}{2}} = -\frac{4}{\pi}$

$y - 2 = -\frac{4}{\pi}\left(x - \frac{\pi}{2}\right)$

At B, $x = -\frac{\pi}{2}$, $y = 2$

$\frac{dy}{dx} = -\frac{2}{-\frac{\pi}{2}} = \frac{4}{\pi}$

$y - 2 = \frac{4}{\pi}\left(x + \frac{\pi}{2}\right)$

$-\cancel{\frac{4}{\pi}}\left(x - \frac{\pi}{2}\right) = \cancel{\frac{4}{\pi}}\left(x + \frac{\pi}{2}\right)$

$2x = 0$

$x = 0$

$y - 2 = -\frac{4}{\pi}\left(0 - \frac{\pi}{2}\right)$

$y - 2 = 2$ $\quad \therefore y = 4$ $\quad P(0, 4)$

At Q, $y = 0$

$0 - 2 = -\frac{4}{\pi}\left(x - \frac{\pi}{2}\right)$

$\frac{\pi}{2} = x - \frac{\pi}{2}$

$x = \pi$

$Q(\pi, 0)$

At R, $y = 0$

$0 - 2 = \frac{4}{\pi}\left(x + \frac{\pi}{2}\right)$

$-\frac{\pi}{2} = x + \frac{\pi}{2}$

$x = -\pi$

$R(-\pi, 0)$

Area $= \frac{1}{2}bh = \frac{1}{2}(2\pi)4 = \boxed{4\pi}$ units squared.

Q8

8

Show that

$$\frac{d}{dx}\left[a^{x^k}\right] = ka^{x^k}x^{k-1}\ln a$$

where a and k are constants.

[3]

Let $y = a^{x^k}$

$\ln y = \ln a^{x^k}$

$\ln y = x^k \ln a$

$\frac{d}{dx}$ $\quad \frac{1}{y}\frac{dy}{dx} = kx^{k-1}\ln a$

$\frac{dy}{dx} = ykx^{k-1}\ln a$

Sub $y = a^{x^k}$

$$\boxed{\frac{d\left(a^{x^k}\right)}{dx} = ka^{x^k}x^{k-1}\ln a}$$